Poisson Brackets and Canonical Transformations

a) Poisson Brackets

- Fundamental notion of Hamiltonian Mechanics

i.e.:

- phase volume conservation
- incompressibility of phase space flow

→ Liouville's Thm.

i.e. $\underline{V}_\Pi = (\dot{q}_i, \dot{p}_i)$

$$\underline{\nabla}_\Pi \cdot \underline{V}_\Pi = \frac{\partial}{\partial q_i}\dot{q}_i + \frac{\partial}{\partial p_i}\dot{p}_i$$

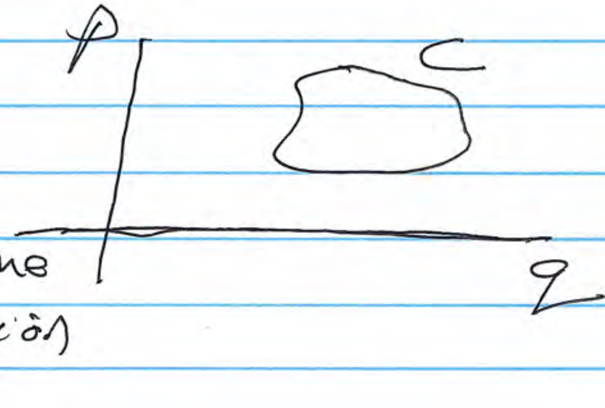
$$= \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(\frac{-\partial H}{\partial q_i}\right) = 0$$

equivalent to:

$$\int_C dp_i\, dq_i = \text{const.}$$

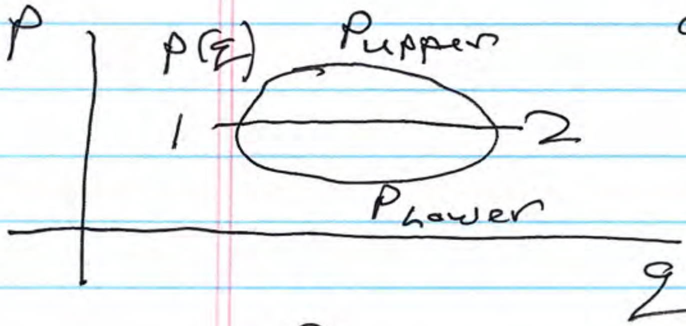
↓ area within C

↓ phase volume conservation



Now $\int dp_i\, dq_i = \oint_C p_i\, dq_i$

$\downarrow$ circulation about C in phase space



$$\underset{\text{enclosed area}}{A} \equiv \int_1^2 p_u(q)\, dq - \int_1^2 p_L(q)\, dq$$

$$= \int_1^2 p_u(q)\, dq + \int_2^1 p_L(q)\, dq$$

$$= \underset{\text{circulation}}{\oint p\, dq}$$

N.B.: Liouville Thm. analogous to Kelvin circulation theorem for isentropic fluids.

Kelvin Thm.

i.e. $\underset{\text{circulation}}{\Gamma} = \oint_C \underline{u} \cdot d\ell = \text{const}$

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v} \; \cancel{-\nu\nabla^2 v} = -\frac{\nabla p}{\rho}.$$

incompressible: $\rho = \text{const}$

$$\frac{\nabla p}{\rho} = \nabla(p/\rho)$$

isentropic:

$$dE = T\,ds - p\,dV$$

$$\underset{\text{enthalpy}}{dw} = T\,ds + V\,dp$$

$$ds = 0, \quad V = 1/\rho$$

$\Rightarrow$

$$dw = \frac{dp}{\rho}$$

$\therefore$

$$\frac{d\underline{v}}{dt} = -\nabla w$$

$$\frac{d}{dt}\Gamma = \oint \frac{d\underline{v}}{dt}\cdot d\ell + \oint \underline{v}\cdot\frac{d\ell}{dt}$$

$$= \oint -\underline{\nabla} w \cdot d\underline{\ell} + \int \underline{v}\cdot d\underline{v}$$

$$= 0 + 0$$

$$\Rightarrow \boxed{\Gamma = \int \underline{v}\cdot d\underline{\ell} = \text{const} \quad \text{is Kelvin Thm.}}$$

Point: Circulations $\left\{\begin{array}{l}\oint p\,dq \\ \oint \underline{v}\cdot d\underline{\ell}\end{array}\right.$ are

dynamically conserved quantities in fluid flow (phase space or otherwise).

Now, Liouville Thm $\Rightarrow$ for any $A(q, p, t)$:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{\partial}{\partial q_i}\left(\frac{dq_i}{dt}A\right) + \frac{\partial}{\partial p_i}\left(\frac{dp_i}{dt}A\right) = 0$$

$$\left(\text{i.e.} \quad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0\right)$$

as $\quad \dfrac{\partial \dot{q}}{\partial q} + \dfrac{\partial \dot{p}}{\partial p} = 0$

$$= \frac{\partial A}{\partial t} + \dot{q}_i\frac{\partial A}{\partial q_i} + \dot{p}_i\frac{\partial A}{\partial p_i} = 0$$

and using HEOM:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \left(\frac{\partial H}{\partial p_i} \frac{\partial A}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial A}{\partial p_i} \right)$$

$$\equiv \frac{\partial A}{\partial t} + \{A, H\}$$

$$\boxed{\begin{array}{l} \{A, H\} \rightarrow \{A, B\} \equiv \text{Poisson Bracket} \\ \{A, B\} = \dfrac{\partial B}{\partial p_i} \dfrac{\partial A}{\partial q_i} - \dfrac{\partial B}{\partial q_i} \dfrac{\partial A}{\partial p_i} \end{array}}$$

→ P.B. is notational shorthand

→ P.B. defines operator relation, specifically a <u>non-commutative Lie Algebra</u>.

→ Bracket Properties

① $\{f, g\} = -\{g, f\}$ (anti-commutativity)

② $\{f + g, h\} = \{f, h\} + \{g, h\}$ (distributive)

③ $\{fg, h\} = f\{g, h\} + g\{f, h\}$

(associative – follows from derivative of product)

④ $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$
$= 0$

Jacobi Identity

→ see L&L for proof
(not instructive)
→ akin cross product rule.

Some key points:

a.) if $\{A, H\} = 0 \Rightarrow dA/dt = 0$

$\boxed{\text{with } A = A(q, p) \quad \text{indep. } t \quad (\text{i.e. } \partial A/\partial t = 0)}$

$\Rightarrow$ A is COM

b.) $\{A, H\} = 0$

$\{B, H\} = 0$

$\Rightarrow$ Jacobi identity:

$\{A, \{B, H\}\} + \{B, \{H, A\}\}$
$+ \{H, \{A, B\}\} = 0$

$\Leftarrow$

so $\{H, \{A, B\}\} = 0$

$\Rightarrow \{A, B\}$ is IOM.

c.) in particular: (Important)

$$\{q_i, q_j\} = 0 \quad , \quad \{p_i, p_j\} = 0$$

$$\{p_i, q_j\} = \delta_{i,j} \quad \Rightarrow \quad [p_i, q_j] = -i\hbar\,\delta_{i,j} \text{ in QM}$$

N.B.:

- Poisson bracket is notational short-hand

but

- also encapsulates key relationships of generalized coordinates.

b) Canonical Transformations

$\Rightarrow$ in general, seek how transform:

$$\left.\begin{matrix} p_i \\ q_i \end{matrix}\right\} \Rightarrow \left.\begin{matrix} P_i \\ Q_i \end{matrix}\right\}$$
"old" "new"

useful for:

$\Rightarrow$ technical aspects of problem — i.e. change of variables

$\Rightarrow$ writing in simplest form i.e. action-angle.

s/t Hamiltonian structure preserved

i.e. if $\quad \dot{p}_i = -\dfrac{\partial H}{\partial q_i}, \quad \dot{q}_i = \dfrac{\partial H}{\partial p_i}$

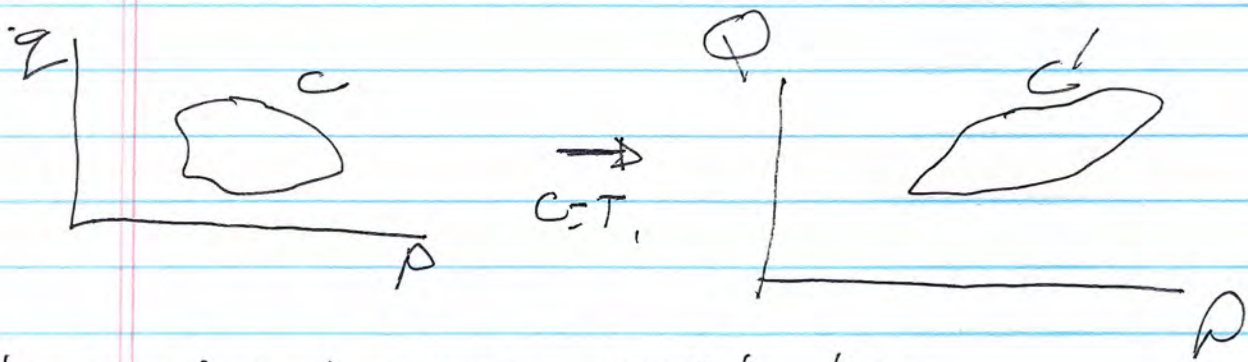
then $\quad \dot{P}_i = -\dfrac{\partial H'}{\partial Q_i}, \quad \dot{Q}_i = \dfrac{\partial H'}{\partial P_i}$.

H' is new Hamiltonian.

Clearly such a transformation must:

① – preserve phase volume
② – preserve bracket relations

①



but enclosed areas must be equal

$$\int_{A_c} dp_i\, dq_i = \int_{A_{c'}} dP_i\, dQ_i$$

so, must have:

$$\frac{\partial(P_i, Q_i)}{\partial(p_i, q_i)} = 1$$

↓
Jacobian of transformation

N.B. {Only require constant to re-scale variables. No loss generality to take as $\underset{\sim}{\text{unity}}$.}

trivial example:

- how transform from q, p to $Q = q(t+\delta t)$, $P = p(t+\delta t)$?

i.e

old	New
p	$P = p(t+\delta t)$
q	$Q = q(t+\delta t)$

Obviously, use Hamiltonian EOMs.

$$P = p(t+\delta t) = p(t) + \delta t\,(dp/dt) \qquad \text{to } O(\delta t)$$

$$= p(t) - \delta t\,(\partial H/\partial q)$$

$$Q = q(t+\delta t) = q(t) + \delta t\,(dq/dt)$$

$$= q(t) + \delta t\,(\partial H/\partial p)$$

H generates transform. Is it canonical?

$$\frac{\partial(P, Q)}{\partial(p, q)} = \begin{vmatrix} \dfrac{\partial P}{\partial p}, & \dfrac{\partial P}{\partial q} \\[2ex] \dfrac{\partial Q}{\partial p}, & \dfrac{\partial Q}{\partial q} \end{vmatrix}$$

$$= \begin{vmatrix} 1 - \delta t \dfrac{\partial^2 H}{\partial p \partial q}, & -\delta t \left(\dfrac{\partial^2 H}{\partial q^2}\right) \\[2ex] \delta t \left(\dfrac{\partial^2 H}{\partial p^2}\right), & 1 + \delta t \dfrac{\partial^2 H}{\partial p \partial q} \end{vmatrix}$$

$$= 1 + \delta t \left(\cancel{\frac{\partial^2 H}{\partial p \partial q}} - \frac{\partial^2 H}{\partial p \partial q} \right)$$

$$+ \delta t^2 \left[\left(\frac{\partial^2 H}{\partial q^2}\right)\left(\frac{\partial^2 H}{\partial p^2}\right) - \left(\frac{\partial^2 H}{\partial p \partial q}\right)^2 \right]$$

$$= 1 + \delta t^2 \{ \text{Gaussian Curvature } H \}$$

so, after expansion to δt,

$$\frac{\partial(P, Q)}{\partial(p, q)} = 1 \quad , \quad \text{to } O(dt^2)$$

i.e. - have shown phase volume conservation to order of calculation

- transformation is canonical.

i.e. H 'generates' canonical transformation $q(t), p(t) \Rightarrow q(t+dt), p(t+dt)$

$\Rightarrow$ Can view H. E.O.Ms as sequence of canonical transformations

$\Rightarrow$ How to Transform Canonically ?

- generally, seek transformation

$$\underset{\text{old}}{p, q} \rightarrow \underset{\text{new}}{P, Q}$$

where have:

- 2 independent variables + Gen. Fctn.
- 2 dependent variables

Point of C.-T. is that preservation of
of Hamiltonian structure immediately
defines dependent variables from independent

4 cases

	independent	dependent
1)	q, Q	p, P
2)	q, P	Q, p
3)	p, Q	q, P
4)	p, P	q, Q

Case I) q, Q independent

p, P tbd ?

Constraint: Liouville Thm:

$$\int dp\,dq = \int dP\,dQ$$

or

$$\oint_C p\,dq = \oint_{C'} P\,dQ$$

(Stokes Thm.)

as $\quad \omega' = \omega'(\epsilon) \Rightarrow$

$$\oint \left[p\,dq - P\,dQ \right]$$

where $\quad p = p(q, Q)$
$\qquad\quad\; P = P(q, Q)$

Now, $\quad 0 = \oint dF_1$

$F_1 = F_1(q, Q) \Rightarrow$ generating function given or tbd.

so

n.b. phase volume cons $\Rightarrow$ 2 variables but one fctn

$$0 = \int dF_1$$

$$= \oint \left(\frac{\partial F_1}{\partial q} dq + \frac{\partial F_1}{\partial Q} dQ \right)$$

$\Rightarrow$

$$\oint \left[p(q, Q)\, dq - P(q, Q)\, dQ \right]$$

$$= \oint \left[\frac{\partial F_1}{\partial q} dq + \frac{\partial F_1}{\partial Q} dQ \right]$$

so

so $$\left.\begin{aligned} p(q,Q) &= \partial F_1/\partial q \\ P(q,Q) &= -\partial F_1/\partial Q \end{aligned}\right\}$$ specify dependent

Example:

$$F_1 = qQ$$

$$p = \partial F_1/\partial q = Q$$

$$P = -\partial F_1/\partial Q = -q$$

old	new
p, q	$P = -q$, $Q = p$

$\left.\begin{aligned} P &= -q \\ Q &= p \end{aligned}\right\}$ illustrates interchangeability of position, momentum in Hamiltonian systems

$\therefore$ $$\boxed{\begin{aligned} &F_1 = F_1(q,Q) = qQ \\ &\Rightarrow \text{interchange/flip} \end{aligned}}$$

Case 2 $\quad q, P$ independent
Q, p ?

Now, conservation of phase volume:

$$\oint \left(p\,dq - P\,dQ \right)$$

but indep q, P ?!

Legendre transform:

$$\oint d(PQ) = 0$$

$$\oint P\,dQ + Q\,dP = 0$$

$$\oint P\,dQ = -\oint Q\,dP$$

$\therefore$

$$\oint (p\,dq + Q\,dP) = 0$$

$$= \oint dF_2$$

$$= \oint \left(\frac{\partial F_2}{\partial q} dq + \frac{\partial F_2}{\partial P} dP \right)$$

so

$$p = \partial F_2 / \partial q$$

$$Q = \partial F_2 / \partial P$$

Example:

- important

Change-of-variable / contact

i.e. $Q = Q(q) = f(q)$

$$F_2 = f(q)\,P$$

$$\therefore \quad p = \frac{\partial F_2}{\partial q} = \frac{\partial f}{\partial q} P$$

$$Q = \frac{\partial F_2}{\partial P} = f(q)$$

old	new
q, p	$Q = f(q)$
	$P = p / \partial f / \partial q$

$$\int dq\,dp = \int dQ\,dP$$

$$= \int \cancel{f'}\,dq\,\frac{dp}{\cancel{f'}} = \int dq\,dp \ \checkmark.$$

– trivial : identify

$$f(q) = q$$

$$F_2 = qP$$

Case 3.) P, Q independent

Now need Legendre transform
$p\,dq \to q\,dp$

$$\oint [p\,dq - P\,dQ] = 0$$

but $\oint d\,(pq) = 0$

$$\oint p\,dq = -\oint q\,dp$$

so taking out sign:

$$\oint [q\,dp + P\,dQ] = 0 = \oint dF_3$$

$$= \oint \left(\frac{\partial F_3}{\partial p} dp + \frac{\partial F_3}{\partial Q} dQ \right)$$

$$q = \partial F_3 / \partial p$$

$$P = \partial F_3 / \partial Q$$

Example

— Momentum C-O-V / contact

i.e. $P = P(p) =$ [redacted] $g(p)$

then obviously:

$$dp\,dq \stackrel{?}{=} dP\,dQ$$

$$= g'(p)\,dp\,dQ$$

so $dQ = dq/g'(p)$ to conserve volume.

Now, take:

$$F_3 = Q\,g(p)$$ [redacted]

so $q = \partial F_3/\partial p = Q$ [redacted] $g'(p)$

$$P = \partial F_3/\partial Q = g(p)$$

Old	New
p	$Q = q/g'(p)$
q	$P = g(p)$

$$dp\,dq = dQ\,dP = \frac{dq}{\cancel{g'(p)}}\,\cancel{g'(p)}\,dp \quad ✓$$

N.B. Natural symmetry:

position C-O-V: $F_2 = P\,f(q)$
trivial: $f(q) = q \rightarrow$ identity

momentum C-O-V: $F_3 = Q\,g(p)$
trivial: $g(p) = p \rightarrow$ identity

Case 4) p, P independent.

Now need 2 Legendre transformations:

$$\oint p\,dq = -\oint q\,dp$$

$$\oint P\,dQ = -\oint Q\,dP$$

so

$$\oint \left[q\,dp - Q\,dP \right] = \oint dF_4$$

$$= \oint \left[\frac{\partial F_4}{\partial p} dp + \frac{\partial F_4}{\partial P} dP \right] = 0$$

$$q = \frac{\partial F_4}{\partial p}$$

$$-Q = \frac{\partial F_4}{\partial P}$$

Application: $F_4 = pP$ $\left\{ \text{Flip/ Interchange} \right.$

then:
$$q = \frac{\partial F_4}{\partial p} = P$$
$$Q = -\frac{\partial F_4}{\partial P} = -p$$

$\underbrace{\text{old}}$ p, q $\underbrace{\text{new}}$ $P = q, \quad Q = -p$

Note:

→ can get flip via

$$F_1 = qQ, \text{ or, } F_4 = pP$$

<u>but</u> independent variables different

→ can get identity / C-O-V via

$$F_2 = P f(q), \text{ or, } F_3 = Q g(p)$$

but independent variables different.

N.B.:

- 4 cases

- 2 flips
- 2 identities
- 2 c-o-v

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ indeps. differ.

Special Case

Orthogonal Transformation

seek $\displaystyle Q_i = \sum_k \underbrace{a_{ik}}_{\text{rotation matrix}} q_k$ rotation

∴ for o-T:

$$F = Q(q)\, p \rightarrow \sum_i p_i Q_i$$

$$= \sum_{i,k} p_i\, a_{ik}\, q_k$$

⟹

$$p_i = \frac{\partial F_2}{\partial q_i} = \sum_k p_k\, a_{ki}$$

$$Q_i = \frac{\partial F_2}{\partial p_i} = \sum_k a_{ik}\, q_k$$

Now, expect since $q \to Q$ rotation $\Rightarrow$
$p \to P$ rotation

check:

$$p_k = \sum_i P_i a_{ik}$$

$$a_{jk} p_k = \sum_i P_i a_{ik} a_{jk}$$

$$= \sum_i P_i \delta_{ij} \qquad a_{jk} \text{ is rotation}$$

$$\therefore \quad P_j = \sum_k p_k a_{jk} \quad \checkmark$$

$\rightarrow$ Harmonic Oscillator

take: $F = F_1 = F_1(q, Q)$

$$= \frac{m}{2} \omega q^2 \cot Q$$

origin ?

$$p = \partial F / \partial q = m\omega q \cot Q$$

$$P = -\partial F / \partial Q = \frac{m\omega}{2} q^2 \csc^2 Q$$

$$= \frac{m\omega}{2} q^2 / \sin^2 Q$$

Now, interesting to express old in terms new:

from P:

$$q = \left(\frac{2P}{m\omega}\right)^{1/2} \sin Q$$

$$p = m\omega q \cot Q = m\omega \left(\frac{2P}{m\omega}\right)^{1/2} \sin Q \frac{\cos Q}{\sin Q}$$

$$= \left(2Pm\omega\right)^{1/2} \cos Q$$

then

$$H = \frac{p^3}{2m} + \frac{1}{2} m\omega^3 q^2$$

$$= \frac{1}{2\cancel{m}}\left(2P\cancel{m}\omega\right)\cos^3 Q + \frac{\cancel{m}}{\cancel{2}}\omega^{\cancel{2}}\frac{\cancel{2}P}{\cancel{m\omega}}\sin^2 Q$$

$$= \omega P$$

$$H = \omega P$$

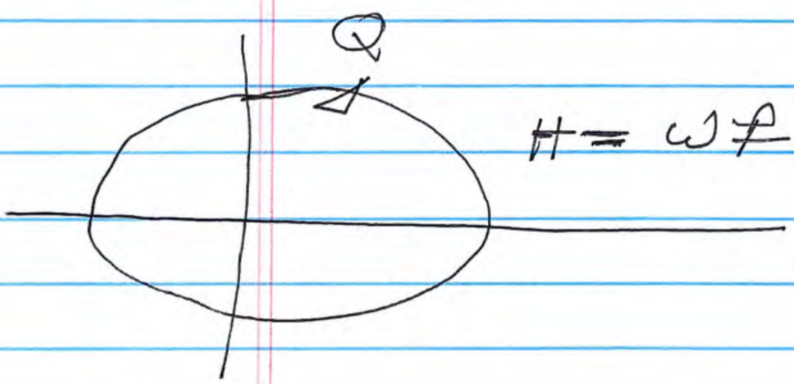
n.b. $P = \frac{H}{\omega} = \frac{E}{\omega}$

action is new momentum.

then $\dot{Q} = \dfrac{\partial H}{\partial P} = \omega$

$$Q = \omega t + Q_0$$

new position is angle:



this brings us to Action - Angle Variables

→ What of the Hamiltonian?

Consider 2 cases:

(a) time independent

(b) time dependent

(a) for time independent case, simply change variables:

$$H' = H'(P, Q) = H(p(P,Q), q(P,Q))$$

need verify:

given: $\dot{p} = -\partial H/\partial q$, $\dot{q} = \partial H/\partial p$ $\Rightarrow$ $\dot{Q} = \partial H'/\partial P$, $\dot{P} = -\partial H'/\partial Q$

Check: $\dot{Q} = \partial H'/\partial P$

$$\dot{Q} \stackrel{?}{=} \frac{\partial H'}{\partial P} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p}$$

$$= \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q}$$

now $H = H(q(Q,P), p(Q,P))$

$$\dot{Q} = \frac{\partial Q}{\partial q}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\right)$$

$$- \frac{\partial Q}{\partial p}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\right)$$

re-grouping:

$$\dot{Q} = \frac{\partial H}{\partial Q}\overbrace{\left(\frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p}\right)}^{0}$$

$$+ \frac{\partial H}{\partial P}\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right)$$

but $\frac{\partial(Q,P)}{\partial(q,p)} = 1$, by definition of C-T

$\therefore$

$$\dot{Q} = \frac{\partial H}{\partial P} = \frac{\partial}{\partial P} H\left(q(Q,P), p(Q,P)\right)$$

$$= \frac{\partial H'}{\partial P}$$

similarly, can show: $\dot{P} = -\frac{\partial H'}{\partial Q}$.

(b) What of time dependent (i.e. explicit) problems?

Now:

- trajectory in old and new variables must satisfy Hamilton's principle.
- Liouville Thm. applies

i.e.

$$\delta \int (p\,dq - H\,dt) = 0$$

$$\delta \int (P\,dQ - H'\,dt) = 0$$

so

$$\int [p\,dq - H\,dt] = \int [P\,dQ - H'\,dt + \underbrace{dF}_{\text{generating fctn.}}]$$

and

$$\int p\,dq = \int P\,dQ$$

$$\Rightarrow \boxed{H' = H + \partial F/\partial t}$$

where F is generating function